Representation Theory Exercise Sheet 3

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MATH0073

These questions cover roughly the first two weeks of lectures and are grouped by the relevant sections. The questions with boxes, e.g. [8.], are to be handed in during the lecture on Friday week 6 (28/02/2020). This excludes any starred parts, which are non-assessed.

1 Section 8

1. Verify that for matrices $\operatorname{tr}(AB) \neq \operatorname{tr}(A)\operatorname{tr}(B)$, in general. Show that characters are not, in general, group homomorphisms. Find a counter-example to the claim: characters are determined by their values on a generating set (examples exist for S_3). (This is different to what we have seen for representations and would not be possible if they were group homomorphisms).

2. It is a fact that the characteristic polynomial of an $(n \times n)$-matrix A is determined by the set of $\operatorname{tr}(A^k)$ as $k > 0$. Verify this for (2×2)-matrices. Explain why this means that the character of a representation ρ also encodes the characteristic polynomials of $\rho(g)$ for all g.

3. Repeat Example 8.8 for the group A_4.

4. Calculate the inner product of an arbitrary character χ and the character of the regular representation $\chi_{k[G]}$. Explain why this fits with Corollary 5.15 i).

[5.] By considering the character table of A_4 or otherwise, prove that there are finite groups which are not subgroups of $\mathrm{GL}_2(\mathbb{C})$.

6. (a) Calculate the conjugacy classes of D_{12}, the dihedral group of order 12.

   (b) Find a $C_2 \times C_2$-quotient of D_{12}. Show that D_{12} has a normal subgroup of order 3 and a quotient isomorphic to $C_2 \times C_2$.

   (c) Find the number of complex irreducible representations and their dimensions. Give the Artin–Wedderburn decomposition of $\mathbb{C}[D_{12}]$. Find all the one-dimensional characters.

   (d) Calculate the matrices of the two-dimensional representation of D_{12} given by its action on a hexagon (it's best to choose your basis to be neighbouring corners). Calculate its character ρ. By calculating $\langle \rho, \rho \rangle$, show that ρ is irreducible.

   (e) Complete that character table of D_{12}.

[7.] (a) Describe the conjugacy classes of S_4 and thus deduce the number of isomorphism classes of irreducible representations of S_4 over $\mathbb{C}$.

   (b) Show that there is a "sign" map $S_4 \twoheadrightarrow C_2$. Find two one-dimensional representations of S_4.

(c) Find the dimensions of the irreducible $\mathbb{C}$-representations of S_4. Use this to calculate the Artin–Wedderburn decomposition of $\mathbb{C}[S_4]$.

(d) Calculate the character of the three-dimensional representation ρ' of Example 8.8. Using tensor products or otherwise, find the character of the other three-dimensional irreducible.

(e) Complete the character table of S_4.

8. (a) Do Exercise 7 from Sections 6+7.

(b) Now, start from the character table of S_4 given in the previous exercise. Which of the irreducible representations are faithful? For those that aren't faithful find their kernels.

(c) Let ψ denote the irreducible representation of dimension 2. Using characters or otherwise, decompose $\psi^{\otimes 2}$ as a direct sum of irreducible representations. Calculate $\psi^{\otimes 3}$ by tensoring by ψ your decomposition of $\psi^{\otimes 2}$. Similarly, show that there is not any $n \in \mathbb{N}$ for which either of the two three-dimensional representations are a summand of $\psi^{\otimes n}$.

(d) Let ρ be your choice of three-dimensional irreducible. Decompose $\rho^{\otimes 2}$.

(e) Show that every irreducible representation of S_4 is a summand of $\rho^{\otimes n}$ for some choice of n.

(f) Use that the regular representation is injective to give an alternative proof that ρ must be faithful.

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